Virtual finite quotients of finitely generated groups

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Abstract

If G is a semidirect product $N \rtimes H$ with N finitely generated then G has the property that every finite group is a quotient of some finite index subgroup of G if and only if one of N and H has this property. This has applications to 3-manifolds and to cyclically presented groups, for instance for any fibred hyperbolic 3-manifold M and any finite simple group S, there is a cyclic cover of M whose fundamental group surjects to S. We also give a short proof of the residual finiteness of ascending HNN extensions of finite rank free groups when the induced map on homology is injective.

1 Introduction

One means of studying a finitely generated group G is to examine the set $\mathcal{F}(G)$ consisting of the finite quotients of G, as is done when taking the profinite completion of G. Even if G is also a residually finite group, this might not give us the full picture. For instance it is unknown whether the following is true (this is problem (F14) in [17]): if there is $n \geq 2$ such that the residually finite, finitely generated group G has $\mathcal{F}(G)$ consisting of all n-generator finite groups, then G is isomorphic to the free group F_n of rank n.

If a finitely generated group G has many finite quotients then it has many finite index subgroups too and we can consider $\mathcal{F}(H)$ for any H of finite index in G. It is the case that $\mathcal{F}(G)$ and $\mathcal{F}(H)$ might look rather different, for instance if G is a perfect group (one which is equal to its commutator subgroup $[G, G]$) then there will be no non-trivial p-groups in $\mathcal{F}(G)$ but there could be a complex collection of p-groups in $\mathcal{F}(H)$ for many primes p. This can happen for the fundamental group of a closed hyperbolic 3-manifold.

In this paper we are interested in the question of which finitely generated groups G have the property that the union of $\mathcal{F}(H)$ over the finite index subgroups H of G consists of all finite groups. It is clear that there are such groups, for instance non abelian free groups or anything that surjects onto one of these groups. Some other examples were given in [14] where it was shown that this property holds for any finitely generated LERF group (one where every finitely generated subgroup is the intersection of finite index subgroups) containing a non abelian free group. They call our property "having every finite group as a virtual quotient". Moreover various consequences were given in [15], which collects together a large number of results on subgroup growth. In Chapter 3 of this book it is mentioned that our property, here called "having every finite group as an upper section", holds for finitely generated groups with superexponential subgroup growth, and also with superpolynomial maximal subgroup growth.

We first find many more examples of finitely generated groups having every finite group as a virtual quotient by showing in Section 2 that if $S \leq G$ then the set of finite quotients $\mathcal{F}(S)$ is contained in the union of $\mathcal{F}(H)$, where H varies over all finite index subgroups of G, provided the following holds: whenever K is normal in S with finite index, there exists a finite index subgroup L of G with $L \cap S = K$. This is straightforward if we impose that L is normal in G but our generalisation holds by adapting an argument in [14]. As we expect there to be many more finite index subgroups of G than finite index normal subgroups, we can exploit this in Section 3 where we examine semidirect products $G = N \rtimes H$, with N finitely generated. We show that G has every finite group as a virtual quotient if and only if one of N and H does (but this need not be true if N is infinitely generated). Thus we can build up many groups with this property by taking repeated semidirect products of finitely generated groups as long as we merely ensure that one of the factors has the property.

In Section 4 we observe that if $G = N \rtimes H$ for a finitely generated N which surjects to the finite group F then N is contained in the finite index

subgroup of G which surjects to F as given by Section 2. We present a simple alternative proof of this and apply it to semidirect products of the form $G = N \rtimes \mathbb{Z}$. We have that $\mathcal{F}(N)$ is contained amongst the union of $\mathcal{F}(G_n)$, where the G_n are the finite cyclic covers of G. This has applications for closed or finite volume hyperbolic 3-manifolds M which are fibred, such as every 2-generated finite group (in particular every finite simple group) is a quotient of a cyclic cover of $\pi_1(M)$. Also the fundamental group of any virtually fibred hyperbolic 3-manifold (it is an open question whether all such 3-manifolds are) has every finite group as a virtual quotient.

We look at cyclically presented groups in Section 5. These are formed by taking any word w in the free group F_d and obtaining the group $G_d(w)$ from the d generator d relator presentation where we take the images of w on cyclically permuting the d generators. These groups have appeared a lot in the literature and, as we can regard $w \in F_d$ to be in F_n for n at least d, we have an infinite family $G_n(w)$ of groups on fixing w but varying n. We can then ask which group theoretic properties hold for infinitely many $G_n(w)$ in the family. By adapting the result in Section 4, we show that given any finite list of finite simple groups, there exist infinitely many n such that $G_n(w)$ surjects onto all groups in this list, provided that w comes from a free-by-cyclic word of rank at least 2. This condition is interpreted as follows: there is a canonical finite cyclic extension $H_n = G_n(w) \rtimes C_n$ for the cyclic group C_n of order n and $H_n = \langle x, t | r(x,t), t^n \rangle$ with $r \in F_2$ not depending on n and having 0 exponent sum in t. To say w comes from a free by cyclic word means that $\langle x, t | r(x,t) \rangle$ has kernel equal to a free group (of finite rank) of the homomorphism which is the exponent sum of t. There are many free-by-cyclic words and it is very efficient to check whether this condition holds.

In the last section we consider ascending HNN extensions. The reason for this is that groups of the form $G = N \rtimes_\alpha \mathbb{Z}$ can be formed using an automorphism α of N, whereas an ascending HNN extension $G = N *_\theta$ generalises this by allowing θ to be an injective endomorphism. We might hope that similar results on virtual finite quotients hold in this case as well, however we first have to recognise as shown in [23] that if N is finitely generated and residually finite then $G = N *_\theta$ need not be residually finite, in contrast to semidirect products $G = N \rtimes_\alpha \mathbb{Z}$. We adapt their construction slightly to obtain an example where the only finite quotients of $G = N *_\theta$ are cyclic. However it was shown in [4] using deep results in algebraic geometry that we do have residual finiteness when the base N of $N *_\theta$ is a finitely generated free

group F_r. We finish by presenting an elementary proof of this in a special case: when the map θ induces an injective homomorphism on the abelianisation $F_r/[F_r, F_r]$. The proof generalises to an ascending HNN extension of any finitely generated group N having a prime p such that N is residually finite p and the homomorphism that θ induces on the p-abelianisation $N/N^p[N, N]$ is invertible.

2 Virtual finite images

If a group is generated by n elements then any quotient has this property too, thus no finitely generated group can surject to all finite groups. However we may ask if every finite group is a virtual image of a given finitely generated group G: this means that for any finite group F there is a finite index subgroup H of G (for which we write $H \leq_f G$) with H surjecting to F. For instance the free group F_n of rank n has this property when $n \geq 2$. Other examples are large groups, where G is large if there is a finite index subgroup of G which surjects to F_2. These include surface groups $\pi_1(S_g)$, where S_g is the orientable surface of genus $g \geq 2$, and (non abelian) limit groups. This definition of large comes from [21], where a "large" property of groups is defined to be an abstract group property P such that if H has P and G surjects to H then G has P, and if $H \leq_f G$ then H has P if and only if G has P. It is shown there that if P is a "large" property satisfied by one finitely generated group then any large finitely generated group must have P.

Proposition 2.1 *Having every finite group as a virtual image is a "large" property.*

Proof. We just need to show that if $H \leq_f G$ and G has every finite group as a virtual image then so does H.

Suppose that $[G : H] = n$. Given a finite group F, we have $L \leq_f G$ with a homomorphism θ from L to the direct product $F \times \ldots \times F$ of k copies of F, where $2^k > n$. Now if $A = L \cap H$ then $[L : A] \leq n$ and moreover $\theta(A)$ also has index at most n in $F \times \ldots \times F$. Now consider the projection π_i from $F \times \ldots \times F$ to the ith factor. If $\pi_i(\theta(A)) = F$ for any i then we are done because $A \leq_f H$, but if not we have that $|\pi_i(\theta(A))| \leq |F|/2$. This means that at most half the elements of F appear in the ith coordinate of the image of A under θ. But if this is true for all i then $|\theta(A)| \leq (|F|/2)^k$ so that $\theta(A)$

has index at least 2^k in $F \times \ldots \times F$, which is a contradiction.

$\square$

Thus this implies that a large finitely generated group has every finite group as a virtual image, although we noted already that it is easy to see directly.

This property is also considered in [15] Chapter 3 under the title of having every finite group as an upper section (where a section of G is a quotient H/N of a subgroup H of G, and upper means that N, hence H, has finite index in G). We will use the two phrases interchangeably throughout. Theorem 3.1 of this book states that if a finitely generated group G does not have every finite group as an upper section then G can have at most exponential subgroup growth type, whereas free groups and large groups have superexponential subgroup growth of type n^n. To define these terms, let $a_n(G)$ be the number of index n subgroups of G and $s_n(G) = a_1(G) + \ldots + a_n(G)$. If G is finitely generated then $a_n(G)$ is finite for all $n \in \mathbb{N}$. We say that G has exponential subgroup growth type if there exist $a, b > 0$ such that $s_n(G) \leq e^{an}$ for all large n and $s_n \geq e^{bn}$ for infinitely many n (and more generally growth of type $f(n)$ by replacing e^n with $f(n)$).

Thus having subgroup growth type which is bigger than exponential, meaning that lim sup$(\frac{\log s_n(G)}{n})$ is infinite, is a major restriction as it implies that every finite group is an upper section. However it is also shown in this book that there exist finitely generated groups with exponential or slower subgroup growth type which have every finite group as an upper section. This is done in Chapter 13 Section 2 by considering profinite groups which are the product of various alternating groups and then taking finitely generated dense subgroups.

We can also count subgroups with a specific property, such as being maximal, normal or subnormal. Another related result, which is Theorem 3.5 (i) in the same book, states that if G does not have every finite group as an upper section then G has polynomial maximal subgroup growth. Again the converse is not true, with similar examples verifying this.

Suppose that H is a subgroup of G (with both finitely generated) and H has every finite group as an upper section. If H has infinite index in G then one would not expect that this property would transfer across to G, for instance if G is an infinite simple group containing a non abelian free subgroup then G has no finite images at all except for the trivial group $I = \{e\}$. However there is one obvious situation when this can be done.

Proposition 2.2 *Suppose that $H \leq G$ and for any finite index normal sub-*

group K of H there exists a finite index normal subgroup N of G such that $H \cap N = K$. If H surjects to the finite group F then G has F as a virtual image.

Proof. This is simply because if $K \trianglelefteq_f H$ with $H/K \cong F$ and $H \cap N = K$ for $N \trianglelefteq_f G$ then $NH/N \cong H/K$ with $NH \leq_f G$.

□

Corollary 2.3 *Suppose $H \leq G$ and H has every finite group as an upper section. If for all $M \leq_f H$ and $K \trianglelefteq_f M$ we have $L \trianglelefteq_f G$ such that $L \cap M = K$ then G has every finite group as an upper section.*

Proof. If we have a finite group F and $M \leq_f H$ with $K \trianglelefteq_f M$ such that $M/K \cong F$ then LM also has F as a finite quotient by Proposition 2.2.

□

In [14] Theorem 2.1 states that if $H \leq G$ with H having every finite group as an upper section and G is LERF, that is every finitely generated subgroup of G is an intersection of finite index subgroups, then G also has every finite group as an upper section. However on examining the proof, it seems that the LERF property is rather stronger than required for the result to hold and so we offer a version of this result with essentially the same proof but with a different hypothesis.

Theorem 2.4 *Suppose that $H \leq G$ and for any finite index normal subgroup K of H there exists a finite index subgroup L of G, but not necessarily normal in G, such that $H \cap L = K$. If H surjects to the finite group F then G has F as a virtual image.*

Proof. On being given $K \trianglelefteq_f H$ with $H/K \cong F$ and $L \leq_f G$ with $L \cap H = K$ we let Δ be the intersection of hLh^{-1} over all $h \in H$ (so that if HL is a subgroup of G, which need not be the case, then Δ is the core of L in HL). As Δ is a subgroup, it is invariant under conjugation by its own elements, and by elements of H too as this just permutes the terms in the intersection. Thus Δ is normal in the subgroup generated by Δ and H, which therefore is ΔH. We have $\Delta H/\Delta \cong H/(\Delta \cap H)$ and we now show that $\Delta \cap H = K$: certainly $\Delta \leq L$ so $\Delta \cap H \leq L \cap H$. Conversely we need to show that for any $h \in H$ and $k \in K$ we have $h^{-1}kh \in L$, but this is true because K is normal

in H and is contained in L. Finally $\Delta \leq_f G$ because $L \leq_f G$ implies that $L \cap H \leq_f H$, with the elements of $L \cap H$ conjugating L to itself.

$\square$

Corollary 2.5 *Suppose $H \leq G$ and H has every finite group as an upper section. If for all $M \leq_f H$ and $K \trianglelefteq_f M$ we have $L \leq_f G$ such that $L \cap M = K$ then G has every finite group as an upper section.*

Proof. This is the same proof as Corollary 2.3 but applying Theorem 2.4 instead of Proposition 2.2.

$\square$

The Long-Reid result in [14] follows because if G is LERF and there is a subgroup M of G with $K \trianglelefteq_f M$ such that M/K is the finite group F then we can find $L \leq_f G$ with $L \cap M = K$. This is done by taking coset representatives $e = m_1, m_2, \ldots, m_n$ of K in M and finding $L_i \leq_f G$ for $2 \leq i \leq n$ with $K \leq L_i$ but $m_i \notin L_i$, then intersecting the L_i.

3 Semidirect Products

In order to apply Corollary 2.5, one needs a wide class of groups where not all members are (or are known to be) large or LERF. Semidirect products provide such a class. If we have groups N and H with an homomorphism $\theta : H \to \mathrm{Aut}(N)$ then we can form the semidirect product $G = N \rtimes_\theta H$ with $G/N \cong H$. Given a group G we may be able to see it as an internal semidirect product, by finding subgroups N and H of G, with N normal, where $NH = G$ and $N \cap H = I$. In this case the homomorphism θ is given by the conjugation action of H on N.

One nice feature of semidirect products $G = NH$ is that their subgroup structure is not too complicated. If L is a subgroup of G then it might not be the case that $L = SR$ for $S \leq N$ and $R \leq H$: indeed this is not even true for direct products. However finite index subgroups of a semidirect product are "not too far away" from having this structure.

Proposition 3.1 *Suppose that $G = N \rtimes_\theta H$. Then any finite index subgroup $L \leq_f G$ contains a finite index subgroup of the form $S \rtimes_\theta R$, where $S \leq_f N$ and $R \leq_f H$. Moreover if N is finitely generated and we are given any finite index subgroup $S \leq_f N$ then we can find $L \leq_f G$ with $L \cap N = S$.*

Proof. Given $L \leq_f G$ we have that $S = L \cap N$ has finite index in N and is also normal in L. On setting $R = L \cap H$ we have that S is preserved under conjugation by R so SR is the subgroup generated by S and R with $S \trianglelefteq SR$, thus SR is also a semidirect product. Also SR has finite index in G: for this we can assume that $L \trianglelefteq G$ by replacing L with a smaller finite index subgroup which will only reduce SR. Then on taking left coset representatives n_i for S in N and h_j for R in H, we have that any $g \in G$ is equal to nh for $n \in N$ and $h \in H$, thus also equal to $n_i s h_j r$ for $s \in S, r \in R$. But this is equal to $n_i h_j s' r \in n_i h_j SR$ because here $S \trianglelefteq G$.

Now suppose we have S with $[N : S] = i$ and note that the set $\mathcal{S}_i$ of index i subgroups of N is finite because N is finitely generated. As H acts on $\mathcal{S}_i$ by conjugation, the stabiliser R of S in H has finite index in H and we can form the finite index subgroup $L = S \rtimes_\theta R$ of G where we restrict θ from R to $\mathrm{Aut}(S)$. But if $g \in SR \cap N$, so that we have respective elements s, r, n with $g = sr = n$ then $r \in N \cap R = I$, thus $g \in S$.

$\square$

If $G = N \rtimes H$ and G is finitely generated then so is H as it is a quotient of G. However this need not imply that N is finitely generated, so our main interest will be in semidirect products where H and N (hence G) are finitely generated. If H has every finite group as a virtual quotient then so does G (indeed this applies if H is merely a quotient of G, by the correspondence theorem). However we can now prove the more surprising fact that the same is true with N and H swapped.

Corollary 3.2 *If $G = N \rtimes H$ with N finitely generated, and N has every finite group as a virtual quotient then so does G.*

Proof. We need to show that the conditions of Corollary 2.5 are satisfied, where in the hypothesis H has now become N. Given $M \leq_f N$ and $K \trianglelefteq_f M$, we can find $R \leq_f H$ such that MR is also a semidirect product by the second part of the proof of Proposition 3.1. Now on applying this proposition again to $MR = M \rtimes R$, we obtain $L \leq_f MR \leq_f G$ with $L \cap M = K$ because M is finitely generated too.

$\square$

Notes: (1) We certainly need N to be finitely generated in Corollary 3.2 as the example in [2] is a semidirect product $G = F_\infty \rtimes \mathbb{Z}$ where F_∞ is a free group of infinite rank but the only finite quotients of G, and of its finite

index subgroups, are cyclic.
(2) Although the conditions of Corollary 2.5 are satisfied for semidirect products $N \rtimes H$ when N is finitely generated, we remark that the conditions in Corollary 2.3 need not be. For instance, take $H = F_n$ with $G = F_n \rtimes_\theta \mathbb{Z}$, $M = H$ and K an index 2 subgroup of H. If there is L normal in G with $L \cap H = K$ then $L \cap H$ is the intersection of normal subgroups and so is normal in G too. This means that $tKt^{-1} = K$ where $t \in G$ generates the factor $\mathbb{Z}$, thus forcing $\theta(K) = K$ which need not be the case.

We now say a few words on largeness and LERF of semidirect products as in the statement of Corollary 3.2, so that $G = N \rtimes H$ with both factors finitely generated and N has every finite group as a virtual quotient. First if we have a direct product $G = N \times H$, or if H is finite so that $N \trianglelefteq_f G$, then the corollary says nothing new. For a direct product G, if one factor is large or both are LERF then G is large or LERF respectively. Moreover if H is finite then G is large or LERF if and only if N is large or LERF respectively.

Now suppose that H is not normal in G (which is equivalent to $G = N \rtimes H$ not being the direct product of N and H) but is infinite. Let us take H to be the smallest infinite group $\mathbb{Z}$ and N to be a group known to have all finite groups as virtual images. If N is the free group F_n for $n \geq 2$ then $F_n \rtimes \mathbb{Z}$ need not be LERF in general by [5], and it is known to be large if it contains $\mathbb{Z} \times \mathbb{Z}$ by [6] but otherwise this is open. Similarly if $N = \pi_1(S_g)$ for S_g the closed orientable surface of genus $g \geq 2$ then $\pi_1(S_g) \rtimes \mathbb{Z}$ need not be LERF in general (for instance one can put together two copies of the above example for $F_n \rtimes \mathbb{Z}$ so that the resulting group contains subgroups which are not LERF), and although some groups of this form have been proved large by geometric considerations, the question of whether all of these groups are large is very much open too. Thus Corollary 3.2 tells us that all groups of the form $F_n \rtimes \mathbb{Z}$ or $\pi_1(S_g) \rtimes \mathbb{Z}$ have every finite group as a virtual quotient.

We now look at the reverse situation where $G = N \rtimes H$ and G has every finite group as an upper section, to see what this implies for N or H.

Lemma 3.3 *A group G has every finite group as an upper section if and only if it has infinitely many distinct alternating groups A_n as upper sections.*

Proof. Every finite group F is a subgroup of A_N for some N (and hence for all $n \geq N$): this is clear for S_N and if the resulting subgroup has odd permutations then we can increase N by 2 and add a 2-cycle to the odd elements.

Now for F and N above, suppose that we have $L \leq_f G$ with a surjection θ from L to A_n for some $n \geq N$. As $F \leq A_n$ we can pull it back to get $\theta^{-1}(F) \leq_f L \leq_f G$ with $\theta\theta^{-1}(F) = F$.

□

Theorem 3.4 *If $G = N \rtimes H$ and G has every finite group as an upper section then either N or H does too.*

Proof. We can assume that there is $n_0 \geq 5$ such that for all $n \geq n_0$ the group A_n is not an upper section of H, as otherwise we are done by Lemma 3.3. Now given any n which is at least n_0, we know there is $L \leq_f G$ and a surjection $\theta : L \to A_n$. As $N \trianglelefteq G$ we have that $S = L \cap N \trianglelefteq L$. Thus $\theta(S)$ is normal in A_n, meaning that if we can eliminate $\theta(S) = I$ we obtain $\theta(S) = A_n$, and as $L \leq_f G$ we get $S \leq_f N$ so A_n is an upper section of N and we are done by Lemma 3.3 again.

Now if $\theta(L) = A_n$ but $\theta(S) = I$ we see that θ factors through $L/S \cong LN/N$. But LN/N is a finite index subgroup of $G/N \cong H$, which does not have A_n as an upper section.

□

We can form repeated semidirect products $G = G_1 \rtimes G_2 \rtimes \ldots \rtimes G_n$ for finitely generated groups G_i, where for this to be defined we would need to bracket all terms in some way and provide the appropriate homomorphisms. What Corollary 3.2 and Theorem 3.4 show is that no matter how this is done, G has every finite group as an upper section if and only if at least one of the G_i does. Consequently it could be argued that the semidirect product is behaving like a direct product for this property.

We also have the following.

Corollary 3.5 *If G is a repeated semidirect product of finitely generated groups $G_1, \ldots, G_n$ and G has bigger than exponential subgroup growth type then at least one of the G_i has every finite group as an upper section. Conversely if one of the G_i has bigger than exponential subgroup growth type then G has every finite group as an upper section.*

Proof. The subgroup growth condition implies that G or G_i has every finite group as an upper section, so now repeatedly apply Theorem 3.4 or Corollary 3.2 respectively.

□

We remark that if $G = N \rtimes H$ and N has bigger than exponential subgroup growth then it is not known whether G does. For instance if $N = F_n$, which has subgroup growth of superexponential type, and $H = \mathbb{Z}$ then we have subgroup growth of superexponential type for G if it contains $\mathbb{Z} \times \mathbb{Z}$ because of largeness, but even exponential growth is not known in the other cases. At least we see here that G has the weaker property of having every finite group as an upper section.

4 Cyclic covers of groups and fibred manifolds

If we look back to Theorem 2.4 and assume the conditions are satisfied, where we now replace H with S, we conclude that if $S \leq G$ and S surjects to the finite group F then a finite index subgroup of G surjects to F as well. But if we examine the proof, we see that this subgroup contains S. If we now specialise to the case where $G = N \rtimes H$ for N finitely generated with $N = S$ above then any subgroup L of G which contains N must be of the form $L = N \rtimes (H \cap L)$, because if $x \in H \cap L$ then $nx \in L$ for all $n \in N$.

We give here a quick alternative proof for semidirect products which can be more useful for constructive purposes.

Proposition 4.1 *If $G = N \rtimes H$ for N and H finitely generated and N surjects to the finite group F then there exists $L \leq_f G$ containing N with L surjecting to F.*

Proof. If $K \trianglelefteq_f N$ with $N/K \cong F$ then, as N is finitely generated, we have a finite index characteristic subgroup C of N which is contained in K. Consequently N/C surjects to F too and the conjugation action of H on N descends to one on N/C. As N/C is finite, we take the intersection $S \leq_f H$ of all point stabilisers of this action, so $sns^{-1} = nC$ for all $n \in N$ and $s \in S$. We then let $L = N \rtimes S$ and observe that the surjection from N to F via N/C extends to L by sending all of S to the identity.

□

This result might not be of interest if there is no reason to favour finite index subgroups of G that contain N over other finite index subgroups. However there is one setting, motivated by topology, where these subgroups are given an important rôle. This is when $G = N \rtimes_\alpha \mathbb{Z}$ for α an automorphism of N. If $\mathbb{Z}$ is generated by the element t then we define the **cyclic cover** G_n of G to be the index n subgroup $\langle N, t^n\rangle$. We have that $G_n \trianglelefteq G$; indeed we can think of G_n as being the kernel of the map from G to the cyclic group C_n given by the exponent sum of t modulo n. If N has a presentation $\langle g_1, \ldots, g_k | r_1, r_2, \ldots\rangle$ then a presentation for G would be

$$\langle g_1, \ldots, g_k, t | r_1, r_2, \ldots, tg_1t^{-1} = \alpha(g_1), \ldots, tg_kt^{-1} = \alpha(g_k)\rangle$$

and for G_n we have

$$\langle g_1, \ldots, g_k, s | r_1, r_2, \ldots, sg_1s^{-1} = \alpha^n(g_1), \ldots, sg_ks^{-1} = \alpha^n(g_k)\rangle$$

where $s = t^n$.

In particular all of the G_n are generated by $k+1$ elements. The connection with topology is that if $N = \pi_1(M)$, the fundamental group of a d dimensional manifold M then on taking a homeomorphism h of M, we can form the $d+1$ dimensional manifold which is fibred over the circle S^1 with fibre M using h, and this has fundamental group $N \rtimes_{h_*} \mathbb{Z}$ where h_* is the automorphism of N induced by h. If $d = 2$ then N must be the fundamental group of a surface, and if this surface is compact and orientable then $N = F_n$ for a bounded surface and $\pi_1(S_g)$ if it is closed. In particular this discussion applies to fibred knots, where N is free and the cyclic covers take on particular importance. We see that in the case where $G = N \rtimes \mathbb{Z}$, any finite index subgroup L of G that contains N is a cyclic cover, as if $G = \langle N, t\rangle$ then $L = \langle N, t^n\rangle$ for n the smallest positive integer with $t^n \in L$. Thus if there is a surjection from N to a finite group F then there is also a surjection from a cyclic cover of G to F. Consequently we see all the finite images of N amongst the finite images of the cyclic covers of G. We also have:

Corollary 4.2 *If the finitely generated group N surjects to the finite group F then for any $G = N \rtimes_\alpha \mathbb{Z}$ there are infinitely many cyclic covers of G that surject to F.*

Proof. On looking at the proof of Proposition 4.1 we see that the cyclic cover G_n of G surjects to F provided that the automorphism of N/C induced by

α satisfies $\alpha^n = \mathrm{id}$ (or instead of C any normal subgroup J of N which is contained in K and fixed by α would do). Thus any integer multiple of n works too.

□

This observation has various consequences if we are interested in (non abelian) finite simple images of groups, as by the classification of finite simple groups all of them are 2-generated.

Corollary 4.3 *If the finitely generated group N surjects to the free group F_2 and G is any group of the form $N \rtimes_\alpha \mathbb{Z}$ then for any finite list of finite simple groups $S_1, \ldots, S_l$ there exists infinitely many cyclic covers of G which surject to all of $S_1, \ldots, S_l$.*

Proof. As N surjects to F_2 it consequently surjects to each of the S_i. Applying Corollary 4.2 gives us an integer n_i such that any cyclic cover of G having index 0 modulo n_i surjects to S_i. Thus we can take any multiple of the lowest common multiple of $n_1, \ldots, n_l$.

□

Again applications are provided by compact 3-manifolds which are fibred over the circle, as if the fibre is a surface of negative Euler characteristic then the fundamental group is either (non abelian) free, a closed orientable surface group (of genus at least 2), or a closed non-orientable surface group (of genus $g \geq 3$) with fundamental group having a presentation $\langle x_1, \ldots, x_g | x_1^2 x_2^2 \ldots x_g^2 \rangle$. This also surjects to F_2 unless $g = 3$ (see [16] page 52). In particular if M is an orientable hyperbolic 3-manifold (with or without boundary) that fibres over the circle then any finite simple group is a quotient of a cyclic cover of M.

We now say a few words on what is known about the finite simple images of the fundamental group of a hyperbolic 3-manifold M of finite volume. It is certainly true that M has infinitely many images of type $PSL(2, \mathbb{F})$ for $\mathbb{F}$ a finite field, coming from the fact that $\pi_1(M)$ is a subgroup of $PSL(2, \mathbb{C})$. However a lot less seems to be known about other types. There exist examples M, both closed and with boundary, where $\pi_1(M)$ surjects to F_2 and so all finite simple groups appear. A range of results are obtained in [11] which fixes a finite simple group F and considers the question of whether $\pi_1(M)$ has F as a quotient from a probabilistic point of view. In particular the authors take a genus $g \geq 2$ and define the concept of a manifold M of a

random Heegaard splitting of genus g and complexity L. There are only finitely many of these for fixed g and L so the probability that M has a particular property, such as $\pi_1(M)$ surjecting to a finite group F, is well defined. It is shown in Proposition 6.1 that for any F this probability tends to a limit $p(F,g)$ as L tends to infinity. Moreover when F is any non abelian finite simple group, Theorem 7.1 obtains the limit of $p(F,g)$ as g tends to infinity: in particular it is strictly between 0 and 1.

But if we specialise to a family of simple groups not involving $PSL(2,\mathbb{F})$ then things are less clear: for instance Question 7.6 of this paper asks whether every closed (or finite volume) hyperbolic 3-manifold has a quotient A_n for some $n \geq 5$. Their Theorem 7.7 does state that given $\epsilon > 0$ there is $g_0 > 0$ such that for all $g \geq g_0$ the probability of a manifold M of a random Heegaard splitting of genus g having a quotient A_n for some $n \geq 5$, or even at least k quotients of distinct groups A_n for fixed k, is at least $1-\epsilon$. We can have variants on this question: does every closed (or finite volume) hyperbolic 3-manifold have a quotient A_n for infinitely many n or all but finitely many n? We do not know of a specific example proven not to have either one of these properties.

We did however locate an example of a closed hyperbolic 3-manifold which surjects to all but finitely many, but not all A_n. In [7] the extended $[3,5,3]$ Coxeter group Γ, which is now known to be the fundamental group of the smallest closed non orientable hyperbolic 3-manifold, is studied along with its orientable double cover Γ^+ (the smallest orientable example). Theorem 4.1 in this paper states that for all large n, A_n and S_n are quotients both of Γ and of Γ^+. This is proved by an intricate argument that links together copies of particular permutation representations of each group in order to form transitive permutation representations of arbitrarily large degree. However it is easily checked, using the given presentation and MAGMA or GAP, that Γ^+ does not surject to small A_n. If one wants a 3-manifold, rather than a 3-orbifold, with this property then they show that the group Σ_{60a} with index 60 in Γ^+ is torsion free, thus $\mathbb{H}^3/\Sigma_{60a}$ is a closed orientable hyperbolic 3-manifold. Again the computer tells us that it does not surject to A_n for $n = 5,6,7,10$ (though it does for 8 and 9). As for large n, any homomorphism sending Γ^+ to A_n maps Σ_{60a} to a subgroup of index at most 60, which must be A_n for $n \geq 61$. In particular this observation shows that if a group G surjects to infinitely many A_n, or all but finitely many A_n, then any finite index subgroup has this property too.

Another point of interest in this question is provided by [15] Theorem 3.5

(iii) which states that if a finitely generated group surjects to only finitely many groups from all A_n and S_n then the growth type for the number of maximal subgroups of index n is at most $n^{\sqrt{n}}$. Now for the free group F_2 it is n^n (the same growth type as for all subgroups of index n) and, as maximal subgroups pull back to maximal subgroups under any surjection by the correspondence theorem, any hyperbolic 3-manifold with a surjection from its fundamental group to F_2 must also have this property. Thus if there exist hyperbolic 3-manifolds with only finitely many surjections to A_n and S_n, we would witness two markedly different types of growth of a natural quantity purely within the class of hyperbolic 3-manifolds.

We finish this section by pointing out that if one returns to the property of having every finite group as an upper section then any closed or finite volume hyperbolic 3-manifold which is virtually fibred, that is having a finite cover or equivalently a finite index subgroup which is fibred, has this property by dropping down and using Corollary 3.2. A famous question of Thurston asks whether this is this case for all closed or finite volume hyperbolic 3-manifolds. Having every finite group as an upper section would also follow from [14] Theorem 2.1 if every such 3-manifold had a fundamental group which is LERF. Currently both these questions are open. Another notorious open question concerns word hyperbolic groups and whether they are always residually finite. It is shown in [19] that if this is true then every (non elementary) word hyperbolic group would have infinitely many non abelian finite simple quotients. It does not however imply unconditionally that a given non elementary word hyperbolic group H has non abelian finite simple quotients. If one could find such a group H and prove that it has no (or just finitely many) non abelian finite simple quotients then the above result implies that H has some quotient which is also non elementary word hyperbolic but not residually finite. However let us quote a comment from [11] as a warning for when a computer program has been given a finite presentation but not found any non abelian finite simple quotients. This quote relates to random presentations of deficiency zero, but it could well apply in other situations:

"In any case, for a typical deficiency 0 group that has no quotients among the first few non-abelian simple groups, it is clear that if it has any such quotient, the index must be so astronomically large as to be far beyond brute force computation."

5 Finite images of cyclically presented groups

The literature on cyclically presented groups is extensive, so rather than a full list of papers and summary of results we content ourselves with citing [9] as a very recent paper on the subject: any interested reader could then follow back the references therein. We also outline problems of this area that are most relevant to us here.

Let v be an element (without loss of generality cyclically reduced) of the free group F_n with free generating set $x_0, \ldots, x_{n-1}$. There is a natural automorphism π_n of F_n having order n, obtained from the n-cycle $(1, 2, \ldots, n)$. We define the **cyclically presented group** $G_n(v)$ to be the group obtained from the deficiency zero presentation

$$\langle x_0, \ldots, x_{n-1} | v(x_0, \ldots, x_{n-1}), \pi_n(v(x_0, \ldots, x_{n-1})), \ldots, \pi_n^{n-1}(v(x_0, \ldots, x_{n-1}))\rangle;$$

of course $\pi_n^i(v(x_0, \ldots, x_{n-1})) = v(x_i, \ldots, x_{n-1+i})$ where we always take subscripts modulo n. In particular the action of π_n descends to $G_n(v)$. One then asks, given $n \in \mathbb{N}$ and any such word $v \in F_n$, what properties does $G_n(v)$ have: for instance is it non-trivial, infinite, non abelian or moreover non soluble? Does it contain F_2 or is it even a large group? Some approaches to this question have looked at a range of v and where n is less than some bound. However as a (cyclically reduced) word of F_d is also one in F_n for $n \geq d$, our focus will be to fix $v \in F_d$ and then examine this infinite family of groups over all $n \geq d$. We can then ask: does $G_n(v)$ have any of the above properties for infinitely many n? How about for all but finitely many n?

Before listing what is already known about the above problem, we mention an approach to these questions which is much used and which will be especially suitable for us here. On being given a cyclically reduced $v \in F_d$ and taking letters x and t, we can define the word $w(x, t) = v(x, txt^{-1}, \ldots, t^{d-1}xt^{-(d-1)}) \in F_2$ (which need not be cyclically reduced but it does not matter). If we define the group $H_n(w)$ by the 2 generator 2 relator presentation $\langle x, t | w(x, t), t^n\rangle$ for any $n \in \mathbb{N}$ then it is easily verified that $H_d(w) = G_d(v) \rtimes_{\pi_d} C_d$ for C_d the cyclic group of order d, and this is also true on replacing d throughout with n when $n \geq d$. Thus on being given v, instead of considering the family $G_n(v)$ we can convert v into w and examine the corresponding family $H_n(w)$. We have a straightforward lemma which will be of use.

Lemma 5.1 *(i) Given a cyclically reduced element $v \in F_d$ and the corresponding families of groups $G_n(v)$ and $H_n(w)$ for $n \geq d$, we have that $G_n(v)$*

is non-trivial, infinite, non soluble, contains F_2 or is large if and only if $H_n(w)$ is not cyclic of order n, infinite, non soluble, contains F_2 or is large respectively.
(ii) If there exists n such that $G_n(v)$, respectively $H_n(w)$, has one of the corresponding properties above then this also holds for $G_m(v)$, respectively $H_m(w)$, whenever n divides m.

Proof. (i) Any property which is preserved by finite index subgroups and extensions by finite cyclic groups will apply equally to $G_n(v)$ and $H_n(w)$. As for triviality, if $H_n(w) = C_n$ then we cannot have $|G_n(v)| > 1$, nor can it be infinite.
(ii) The point is that if n divides m then $H_m(w)$ surjects to $H_n(w)$ by adding the relation t^n. By abelianising the presentation for any $H_n(w)$, we see that it must surject to C_n with t mapping to a generator because the exponent sum of t in $w(x,t)$ is zero. Now if $H_m(w) = C_m$ then it is the cyclic group $\langle t | t^m \rangle$, thus $H_n(w)$ is the group $\langle t | t^m, t^n \rangle = C_n$. All other properties mentioned above for $H_n(w)$ are preserved by prequotients, and we can now use Part (i) to transfer these over to $G_n(v)$.

□

Thus we have that possessing any of these properties for some n in a family of the form $\{G_n(v) : n \geq d\}$ or $\{H_n(w) : n \geq d\}$ implies the same for infinitely many n.

Given such a $v \in F_d$ (and corresponding $w(x,t) \in F_2$) we can form the **associated polynomial** $f_v(t) \in \mathbb{Z}[t]$ of degree at most $d-1$ which is defined by $f_v(t) = a_{d-1}t^{d-1} + \ldots + a_1 t + a_0$, where a_i is the exponent sum of the letter x_i in v (in fact this is just the Alexander polynomial of $w(x,t)$, although the latter is a Laurent polynomial which is only defined up to multiplication by $\pm t^j$ for any $j \in \mathbb{Z}$). This polynomial plays an important part in the theory of cyclically presented groups, although of course different words $v_1, v_2 \in F_d$ can have the same polynomial $f_{v_1} = f_{v_2}$; indeed this happens if and only if v_1 and v_2 represent the same element in the abelianisation $\mathbb{Z}^d$ of F_d.

We now give a brief outline of known results in relation to the problems above. First we ask: is there v such that $G_n(v)$ is trivial for infinitely many $n \geq d$? The only examples we know are the obvious ones $v = x_i^{\pm 1}$ for $0 \leq i \leq d-1$, giving $w = t^i x t^{-i}$. In [9] the abelianisation $G_n(v)^{\text{ab}} = G_n(v)/[G_n(v), G_n(v)]$ of $G_n(v)$ is studied, using results in [8]. Let us say that f_v is of cyclotomic type if it is a product of (not necessarily distinct) cyclotomic polynomials $\pm\Phi_m$, up to multiplication by $\pm t^i$. It is shown there

that if f_v is not of cyclotomic type (and $f_v \neq \pm t^i$) then $G_n(v)^{\mathrm{ab}}$ is non trivial for all but finitely many n and if f is of cyclotomic type then $G_n(v)^{\mathrm{ab}}$ is non trivial for infinitely many n. Thus only v with an associated polynomial of cyclotomic type could possibly yield $G_n(v)$ being trivial for infinitely many n and only $\pm t^i$ could do for all n, but we do not know of any examples other than the above in either case. We remark in Example 5.4 that there exist cases where $f_v(t) = \pm t^i$ but which need other methods to show non triviality.

However any hope that there might be equivalent statements or open questions for the other properties can be dispelled because there exist examples where $G_n(v)$ is finite cyclic for all n. For instance Theorem 3 of [22] implies that if we take $v = x_0^\alpha x_1^{-\beta}$ with α and β integers at least 2 which are coprime then $G_n(v)$ is cyclic of order $\alpha^n - \beta^n$. But if we are happy to argue "generically" then we can use a powerful theorem of Ol'shanskiĭ in [18]. This states that if H is a non elementary word hyperbolic group and $g \in H$ is any element of infinite order then the quotient $Q_n(g)$ of H formed by adding the relator g^n is again non elementary word hyperbolic for all but finitely many n. In particular we see that if the 2 generator 1 relator group $H(w) = \langle x, t | w(x,t)\rangle$ is a non elementary word hyperbolic group then $H_n(w)$ will be non elementary word hyperbolic as well for all but finitely many n, thus these groups will certainly be infinite, non soluble and will contain F_2. However we note that we can certainly have $H(w)$ not word hyperbolic but $H_n(w)$ is for all large n, as when performing Dehn filling which is mentioned later. The claim of genericity holds because given a free group F_r of finite rank $r \geq 2$ and any word in F_r of length at most l, the proportion of 1-relator groups thus obtained which are non elementary word hyperbolic tends to 1 as l tends to infinity. (Here the situation is slightly different as our word $w(x,t)$ has exponent sum zero in t, but we can always apply a change of basis to put an arbitrary word in this form.)

There is also a parallel statement for largeness: if H is a finitely generated large group and $g \in H$ is any element then Lackenby shows in [13] that the quotient $Q_n(g)$ formed by adding the relator g^n is also large, but now only for infinitely many n. Note that if we get largeness of $Q_n(g)$ for one value of n then we immediately get largeness for all $Q_{kn}(g)$ with $k \geq 1$ as $Q_{kn}(g)$ surjects to $Q_n(g)$. There is a short proof of this result in [20], but Corollary 3.5 of [3] gives an example of a large torsion free hyperbolic group H and an element $g \in H$ such that $Q_n(g)$ is not large for odd n. Consequently we see that by starting with a large group of the form $H(w) = \langle x, t | w(x,t)\rangle$ with

the exponent sum of t being zero, for which there are lots of examples, then we obtain families $H_n(w)$ and $G_n(v)$ which are large for infinitely many n, but we do not know of examples which are large for all but finitely many n.

However let us now concentrate on the finite quotients of $H_n(v)$ and $G_n(w)$. We might ask under what circumstances do we find examples having non abelian finite simple quotients: this need not be implied by the fact that the group is infinite or non soluble, and is not known to be implied if the group is non elementary word hyperbolic or large. However there is a class of groups where we can use the results in Section 4 to get a positive answer. Suppose that v is any cyclically reduced word in F_d such that the resulting 2 generator 1 relator group $H(w) = \langle x, t | w(x,t) \rangle$ is a semidirect product $F_r \rtimes_\alpha \mathbb{Z}$ of a free group of rank r (for $r \geq 2$) with the integers, and where t generates $\mathbb{Z}$ with $x \in F_r$ and the automorphism α is conjugation by t. If this is the case then let us call w a **free-by-cyclic** word of rank r. In fact it is straightforward and well known to tell if a given w is a free-by-cyclic word: first suppose that the cyclically reduced word $v(x_0, x_1, \ldots, x_{d-1})$ is such that the generator with the smallest index that actually appears in v is x_s, and similarly x_l is the largest, where $0 \leq s \leq l \leq d-1$ and $r = l - s$. The condition which must hold is that x_s appears only once in v, either as itself or as its inverse, and the same for x_l. If so then the associated polynomial $f_w(t)$ is monic at both ends (both the largest non zero coefficient a_l and smallest non zero coefficient a_s are equal to ± 1) and $r = l - s \leq d$. We then have:

Theorem 5.2 *If v is any cyclically reduced word in F_d giving rise to a free-by-cyclic word w of rank r at least 2 then any non abelian finite simple group appears as a quotient of the cyclically presented group $G_n(v)$ for some n, indeed infinitely many n. This is also true if given a finite list of non abelian finite simple groups: there will be infinitely many n such that $G_n(v)$ surjects to all of these groups.*

Proof. This is an application of Corollaries 4.2 and 4.3 with N equal to F_r and G taken to be $H(w) = \langle x, t \rangle = F_r \rtimes_\alpha \mathbb{Z}$. We conclude that infinitely many cyclic covers $\langle F_r, s = t^n \rangle$ of $H(w)$ surject to all of these finite simple groups. Now if we regard the nth cyclic cover as the kernel of the map $\epsilon_n(t) : H(w) \to C_n$ given by the exponential sum of t modulo n, this homomorphism will factor through $H_n(w)$ as we are just adding the relator t^n. Now the kernel of this map from $H_n(w)$ to C_n is $G_n(v)$, thus on taking the nth cyclic cover and setting s to be the identity, the quotient so obtained is

$G_n(v)$. But the maps from our cyclic covers to finite simple groups given in Proposition 4.1 involve setting $s = t^n$ equal to the identity, so they factor through $G_n(v)$.

□

Consequently in Theorem 5.2 we will have that infinitely many $G_n(v)$ are non soluble. However easy examples show that we need not have all but finitely many $G_n(v)$ being non soluble.

Example 5.3 *Let $v \in F_4$ be $x_3x_0^{-1}$, so that $w(x,t) = t^3xt^{-3}x^{-1}$ which is a free by cyclic word of rank 3. Then*

$$G_n(v) = \langle x_0, x_1, \ldots, x_{n-1}, t | x_i = x_{i+3} (0 \leq i \leq n-1) \rangle$$

which is F_3 if 3 divides n but $\mathbb{Z}$ if not.

We remark that free by cyclic words seem to be common but not generic: in [10] it was shown that the proportion of cyclically reduced words in F_2 of length l that give a free by cyclic group has lim sup strictly less than 1 but lim inf strictly bigger than 0 as l tends to infinity. However experimental evidence suggests a limit of about 0.94. (Again the point applies about zero exponent sum.)

We also mention that Thurston's theorem on orbifold Dehn filling can sometimes be used in this context. If we have a free by cyclic group $F_r \rtimes_\alpha \mathbb{Z} = \langle F_r, t \rangle$ for $r \geq 2$ which is the fundamental group of a finite volume orientable hyperbolic 3-manifold M then M is fibred over the circle with fibre an orientable surface with boundary having fundamental group F_r. A necessary but not sufficient condition for this is that there exists a non identity $x \in F_r$ such that some positive power of α sends x to a conjugate of itself. Consequently $\pi_1(M)$ contains $\mathbb{Z} \times \mathbb{Z}$ and is not a word hyperbolic group. Suppose we have $\alpha(x) = x$, which can be achieved by first replacing α with a power $\beta = \alpha^k$, thus moving to a cyclic cover, and then multiplying β by an inner automorphism, preserving the group and the manifold but changing the element $s = t^k$. Then s commutes with x and so both are parabolic elements, corresponding to a torus boundary component. On adding the relation s^n to our group, which corresponds to gluing in a solid torus to this boundary component such that n copies of the loop on this component represented by s are identified with the compressible curve, we obtain an orbifold and Thurston's result tells us that for all sufficiently large n the result is a hyperbolic orbifold. Thus we have that if $H(x,t)$ is a free by cyclic word of

rank $r \geq 2$ that is the fundamental group of a finite volume orientable hyperbolic 3-manifold and t conjugates a non trivial element of the free group to itself then $H_n(w)$ and $G_n(v)$ are infinite and indeed contain non abelian free groups for all sufficiently large n. If the manifold has only one cusp, as would be the case if the free group was of rank 2, then the resulting hyperbolic orbifold is closed so that addition of the relation t^n has turned a non word hyperbolic group into a word hyperbolic group. However the original group is relatively hyperbolic with respect to the boundary elements and the idea of adding a high powered relation stems from Thurston's results in 3 dimensional hyperbolic geometry.

Finally we finish this section with a famous example which shows that looking at finite images cannot give the whole picture.

Example 5.4

Let $v \in F_2$ be $x_1 x_0 x_1^{-1} x_0^{-2}$, so that $w = txt^{-1}xtx^{-1}t^{-1}x^{-2}$. The word v really goes back to an example of Higman, whereas $H(x,t) = \langle x,t | w(x,t)\rangle$ is known as the Baumslag-Gersten group and it was shown in [1] that all finite quotients of $H(x,t)$ are cyclic with x mapping to the identity. This must also be the case for $H_n(w) = \langle x,t|w(x,t),t^n\rangle = G_n(v) \rtimes C_n$ and this property remains true for finite index subgroups: suppose $G \leq_f H$ and all finite index subgroups of H contain its commutator subgroup H'. Then any finite index subgroup L of G is also one of H and $G' \leq H'$ implies that $G' \leq L$. But $G_n(v)$ is a perfect group for all n so has no proper finite index subgroups at all. However Theorem 3 of [22] shows that $G_n(v)$ and $H_n(w)$ are infinite for all $n \geq 4$.

6 Ascending HNN extensions

One generalisation of a semidirect product of the form $N \rtimes_\alpha \mathbb{Z}$ is an ascending HNN extension $N*_\theta$. Whereas α must be an automorphism of N, we only require that $\theta : N \to N$ is an injective homomorphism, not necessarily surjective (although N needs to have proper subgroups isomorphic to itself, namely N is non Hopfian, in order for existence of a θ which is not an automorphism). If $\langle X|R\rangle$ is a presentation for N then, on taking a stable letter t we obtain the presentation $\langle X,t|R, txt^{-1} = \theta(x) \forall x \in X\rangle$.

Ascending HNN extensions sometimes have comparable properties to semidirect products, so we can ask whether N having all finite groups as

virtual images implies the same for $N*_\theta$. In fact we can even ask the same question for any HNN extension in which N is the base. It is certainly not true for non ascending HNN extensions, which is where we have an isomorphism $\theta : A \to B$ of the associated subgroups A and B of N, with both A and B proper subgroups. To see this, let $N = \langle x, y | x^3 = y^2 \rangle$ which is the fundamental group of the trefoil knot, thus has every finite group as a virtual image (for a number of reasons, for instance by Corollary 3.2 as the knot is fibred). But on taking $A = \langle x \rangle$ and $B = \langle y \rangle$ with $\theta(x) = y$ we get

$$N*_\theta = \langle x, y, t | txt^{-1} = y, x^3 = y^2 \rangle$$

which on eliminating y is seen to be the famous non Hopfian Baumslag Solitar group $BS(2,3)$ whose only finite quotients are metabelian.

Unfortunately a similar phenomenon can happen for ascending HNN extensions, as was demonstrated in [23], where the ascending HNN extension $\Gamma = G*_\theta$ of the Grigorchuk group G formed by using the Lysenok extension is shown to have all finite images metabelian: indeed the image of G is shown to be a quotient of $(C_2)^2$. As is similar to the argument in Example 5.4, if a group Γ has every finite image metabelian then the finite residual R_Γ (the intersection of all finite index subgroups) would contain the second derived group Γ''. But as $R_\Gamma = R_\Delta$ for any $\Delta \leq_f \Gamma$, we would have $\Delta'' \leq \Gamma'' \leq R_\Delta$ so all finite images of Δ are metabelian too. Now the Grigorchuk group G is a 2-group so certainly does not have every finite group as a virtual image: only finite 2-groups, which must be nilpotent, can appear here. But G has a much wider range of finite images than the finite index subgroups of Γ: if G had only metabelian finite images then it would be metabelian itself, as G is residually finite so $R_G = I$. However this is not true as G is a finitely generated infinite torsion group. In particular Γ cannot be metabelian and so this paper gives us an example of a non residually finite ascending HNN extension where the base is finitely generated and residually finite. This is in contrast to semidirect products where Malce'ev showed that if N is finitely generated then $N \rtimes H$ is residually finite if both N and H are too. The proof is essentially Proposition 3.1, although we again remind ourselves of the example in [2] showing that this result fails if N is not finitely generated.

This suggests that if we are given a finitely generated residually finite group N which has every finite group as a virtual quotient then it seems unreasonable to expect that an ascending HNN extension $N*_\theta$ will have this property too unless we already know that the extension is residually finite

as well. One case where this has been established is in [4] which shows that ascending HNN extensions of free groups F_r are residually finite. In contrast to the quick proof for semidirect products, this argument is deep and highly non trivial, involving material in algebraic geometry (further use is made of this area, as well as some model theory, to generalise the conclusion to ascending HNN extensions of finitely generated linear groups). Whilst we do not invoke this theorem to establish that ascending HNN extensions of free groups F_r have every finite group as a virtual quotient (which we leave open), in the course of looking for a proof we were able to come up with a considerable simplification of the residually finite result for certain endomorphisms; those that induce an injective map on the abelianisation of F_r.

To provide the necessary background, first note that any ascending HNN extension $\Gamma = G*_\theta$ with stable letter t has an associated homomorphism $\chi : \Gamma \to \mathbb{Z}$ given by the exponent sum of t in an element of Γ. Now Γ is also a semidirect product $K \rtimes \mathbb{Z}$ where $K = \ker(\theta)$ but $K = \cup_{i\in\mathbb{N}} t^{-i}Gt^i$ which is an ascending union, and a strictly ascending union if θ is not surjective (which means that K is not finitely generated). Thus any element not in K is preserved under some homomorphism to a finite cyclic group. Moreover any element in K is conjugate to one in G, so if a conjugate of $g \in G$ is in R_Γ, g will be as well because $R_\Gamma \trianglelefteq \Gamma$.

Consequently if G is residually finite, in order to establish residual finiteness for an ascending HNN extension $\Gamma = G*_\theta$ we need only consider the non identity elements x of G and look for some finite index subgroup $\Delta \leq \Gamma$ with $x \notin \Delta$. We would like to use the fact that we have finite index subgroups H of G with $x \notin H$, for instance we would be done if such an H somehow gave rise to a Δ satisfying $\Delta \cap G = H$. However in the strictly ascending situation, there are severe restrictions on which $H \leq_f G$ are the intersection with G of a finite index subgroup of Γ.

Proposition 6.1 *If $\Gamma = G*_\theta$ is an ascending HNN extension of a group G and $H \leq_f G$ then there exists $\Delta \leq_f \Gamma$ with $\Delta \cap G = H$ if and only if there is $l > 0$ with $\theta^{-l}(H) = H$.*

Proof. Suppose on being given H we have such a Δ. Being of finite index implies there is $l > 0$ with $t^l \in \Delta$. As any $h \in H$ is in Δ, we have that $t^l h t^{-l} = \theta^l(h)$ is in Δ but also in G, so $\theta^l(H) \leq H$, implying $H \leq \theta^{-l}(H)$. Now take $g \in \theta^{-l}(H)$, so that $g \in G$ of course. We have $g = t^{-l}h_0t^l$ for some $h_0 \in H$, and $t^l, h_0 \in \Delta$ implies that g is too, thus $g \in H$.

Conversely it is shown in [6] Proposition 4.3 (iv) by a short but careful argument that if H is any finite index subgroup of G then $\langle H,t\rangle \leq_f \Gamma$. Now, just as for semidirect products over $\mathbb{Z}$, we have cyclic covers $\Gamma_n = \langle G,t^n\rangle$ of Γ which are themselves ascending HNN extensions with $s=t^n$ as stable letter, formed by using the endomorphism θ^n. Thus if we have $H=\theta^{-l}(H)=\theta^{-2l}(H)=\theta^{-3l}(H)=\ldots$ then $\Delta=\langle H,s=t^l\rangle \leq_f \Gamma_l=\langle G,s\rangle \leq_f \Gamma$. It is clear that $H\leq \Delta\cap G$ so let $g\in\Delta\cap G$. As $\theta^l(H)\leq H$, we have that Δ is also an ascending HNN extension with stable letter s, by restricting θ^l to H. This means that any element of Δ can be expressed in the form $s^{-p}hs^q$ for $p,q\geq 0$ and $h\in H$. Now if $g=s^{-p}hs^q$ then we must have $p=q$ as g is in the kernel of the associated homomorphism (which is just restriction to Δ of that for Γ). Thus $\theta^{pl}(g)=s^pgs^{-p}\in H$, meaning that $g\in\theta^{-pl}(H)=H$.

□

As for finding such subgroups which are invariant under pullback by (a power of) θ, it is shown in [6] Theorem 4.4 that if G is finitely generated (which henceforth we will assume) then on repeatedly pulling back H via θ, we obtain $\theta^{-k}(H)=\theta^{-k-l}(H)$ for some $k\geq 0$ and $l>0$. This means that on setting $L=\theta^{-k}(H)$ we have $\theta^{-l}(L)=L$. However it could well be that we find L is all of G anyway. What is required is a good supply of fully invariant subgroups, meaning that $\theta(L)\leq L$ for any endomorphism θ, which implies that L is contained in $\theta^{-1}(L)$.

Now further suppose that L has finite index in G. In this case we would have $L=\theta^{-1}(L)$ if and only if $[G:L]=[G:\theta^{-1}(L)]$. In fact this happens if and only if $\theta(G)L=G$. This follows because the right hand side is equal to $[\theta^{-1}\theta(G):\theta^{-1}(L\cap\theta(G)]$, and as $\theta(G)$ and $L\cap\theta(G)$ are obviously in the image of θ, this index is preserved on removing θ^{-1} to get $[\theta(G):L\cap\theta(G)]=[\theta(G)L:L]$.

Possibilities for these fully invariant subgroups are, given a prime p, the derived p-series and the lower central p-series, both of which intersect in the identity in the case of a free group F_r and have first term $F_r^p[F_r,F_r]$ with quotient $(C_p)^r$.

Theorem 6.2 *If F_r is the free group of rank $r\geq 2$ and θ is an injective endomorphism of F_r then consider the induced homomorphism of abelianisations $\overline{\theta}:\mathbb{Z}^r\to\mathbb{Z}^r$ given by $\theta(x)[F_r,F_r]=\overline{\theta}(x[F_r,F_r])$. If $det(\overline{\theta})\neq 0$ then the ascending HNN extension F_r*_θ is residually finite.*

Proof. Given any prime p, we can consider the endomorphism $\overline{\theta}_p$ of $(C_p)^r$

by reducing $\overline{\theta}$ mod p. If $\det(\overline{\theta}) \neq 0$ when considered as an endomorphism of $\mathbb{Z}^r$ then, by taking a prime p which does not divide $\det(\overline{\theta})$ we have that $\overline{\theta}_p$ is invertible.

Thus in the case where G is the free group F_r, on being given a non identity element x of G we choose a term L_i of the derived or other appropriate p-series for F_r where $x \notin L_i$. Then $\theta(L_i) \leq L_i \trianglelefteq_f G$, allowing us to take the finite index subgroup $S_i = \langle L_i, t\rangle$ of Γ. We are done if we can show $\theta(G)L_i = G$ because then we would have $\theta^{-1}(L_i) = L_i$ by the above, so we can apply Proposition 6.1 to conclude that $S_i \cap G = L_i$ and $x \notin S_i$. Now $\theta(G)L_i = G$ if and only if $\theta(G)L_i/L_i = G/L_i$ but $\theta(G)L_i/L_i$ is the image of $\theta(G)$ under the quotient map q_i from G to G/L_i.

We certainly have that $q_i(\theta(G)) = G/L_i$ if $i = 1$ in which case $L_1 = G^p[G,G]$, since by our assumption on p we have that $q_1\theta(G) = \overline{\theta}_p((C_p)^r)$ is all of $(C_p)^r = G/L_1$. However as G/L_i is a finite p-group, we can utilise the Frattini subgroup (intersection of all maximal subgroups). If this subgroup is finitely generated then a set generates the whole group if and only if it generates the group when quotiented by the Frattini subgroup. Now in the case of a finite p-group P, the Frattini subgroup is $P^p[P,P]$. Thus for any i we have that the Frattini subgroup of $P = G/L_i$ is the image of $G^p[G,G]$ under q_i, so the quotient of G/L_i by this subgroup is $G/(G^p[G,G]L_i)$. But $L_i \leq G^p[G,G] = L_1$ and so the image of $\theta(G)$ in G/L_i is all of G/L_i.

□

We have written out this proof so that it applies in more general situations:

Corollary 6.3 *If G is any finitely generated group which is residually finite p and θ is an injective endomorphism of G then the associated HNN extension $G*_\theta$ is residually finite provided that the induced endomorphism of $G/G^p[G,G]$ is invertible.*

Proof. The residually finite p condition is equivalent to the derived or lower central p-series intersecting in the identity. Now the proof proceeds as before and the invertible condition is used to invoke the Frattini argument at the end.

□

There are a range of groups which are residually finite p, for instance any finitely generated linear group in characteristic 0 is virtually residually finite

p for all but finitely many primes p (again due to Malce'ev), thus we can find examples amongst the finite index subgroups of any such linear group. We remark though that a necessary condition for a non-cyclic finitely generated group G to be residually finite p is that G surjects to $C_p \times C_p$, as otherwise all finite p-images of G are cyclic which would imply in this case that G was too.

Example: The finitely presented ascending HNN extension $\Gamma = G*_\theta$ of the Grigorchuk group $G = \langle a, c, d\rangle$ which is shown to be non residually finite in [23] is formed using the injective endomorphism σ, where $\sigma(a) = aca, \sigma(c) = dc, \sigma(d) = c$. As $G/G^2[G,G] = (C_2)^3$, generated by the images of a, c, d, we see that the induced homomorphism on $G/G^2[G,G]$ is not injective, with ad in the kernel. (If it were then Corollary 6.3 would give us the first example of a finitely presented, residually finite group which is not virtually soluble nor contains F_2. This is because G is a 2-group and residually finite, so residually finite 2.)

In fact we can adjust Γ slightly to come up with a group which is "even less residually finite". Any ascending HNN extension must surject to $\mathbb{Z}$ and so have some finite index subgroups, namely the cyclic covers. However here we have an example where these are all the finite index subgroups, even though the base is finitely generated and residually finite.

Proposition 6.4 *Let G be the Grigorchuk group and $[G,G]$ its commutator subgroup of index 8. Then the only finite index subgroups of the ascending HNN extension $\Delta = [G,G]*_\sigma$, where we restrict σ to $[G,G]$, are the cyclic covers.*

Proof. We have that Δ is a finite index subgroup of $\Gamma = G*_\sigma$ by [6] Proposition 4.3 (iv); in fact it can be checked that Δ has index 4. Moreover it is shown in [23] that in any finite quotient of Γ, the base G maps to an abelian subgroup and so $[G,G]$ maps to the identity. Suppose there exists a finite index normal subgroup N of Δ such that the image of $[G,G]$ is non trivial in Δ/N. Although N need not be normal in Γ, we can find $M \trianglelefteq_f \Gamma$ with $M \leq N$, so that the image of $[G,G]$ is non trivial in Γ/M which is a contradiction.

This means that every finite index normal subgroup of Δ contains $[G,G]$ and hence also the kernel of the associated homomorphism for $\Delta = [G,G]*_\sigma$, thus the only finite quotients of Δ are cyclic and the only finite index subgroups are the cyclic covers.

□

Example: In [12] the injective endomorphism $\theta(a) = b, \theta(b) = a^2$ of the rank two free group $F(a,b)$ is considered. The resulting ascending HNN extension $F(a,b)*_\theta$ is shown to be a 1-relator group $\langle a,t|t^2at^{-2} = a^2\rangle$ which is non linear (by using results of Wehrfritz) but residually finite (using [4]). Here we see the conditions in Theorem 6.2 are satisfied because $\det(\overline{\theta}) = -2$, so we can use any prime but 2 to complete an proof that $F(a,b)*_\theta$ has these properties without recourse to the sophisticated results in [4].

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